Assignment 5

Hand in no. 1, 2, 5, 6 and 10 by October 10, 2019.

1. In a metric space (X,d), its metric ball is the set $\{y \in X : \ d(y,x) < r\}$ where x is the center and r the radius of the ball. May denote it by $B_r(x)$. Draw the unit metric balls centered at the origin with respect to the metrics d_2, d_∞ and d_1 on $\mathbb{R}^2$.

2. Determine the metric ball of radius r in (X,d) where d is the discrete metric, that is, $d(x,y) = 1$ if $x \neq y$.

3. Consider the function Φ defined on $C[a,b]$
$$\Phi(f) = \int_a^b \sqrt{1+f^2(x)}\ dx.$$
Show that it is continuous in $C[a,b]$ under both the supnorm and the L^1-norm.

4. Consider the function Ψ defined on $C[a,b]$ given by $\Psi(f) = f(x_0)$ where $x_0 \in [a,b]$ is fixed. Show that it is continuous in the supnorm but not in the L^1-norm. Suggestion: Produce a sequence $\{f_n\}$ with $\|f_n\|_1 \to 0$ but $f_n(x_0) = 1,\ \forall n$. Ψ is called an evaluation map.

5. Let K be a continuous function defined on $[0,1] \times [0,1]$ and consider the map
$$T(f)(x) = \int_0^1 K(x,y)f(y)dy\ .$$
Show that this map maps $(C[0,1], \|\cdot\|_1)$ to $(C[0,1], \|\cdot\|_\infty)$ continuously.

6. Let A and B be two sets in (X,d) satisfying $d(A,B) > 0$ where
$$d(A,B) \equiv \inf\{d(x,y) : \ (x,y) \in A \times B\}\,.$$
Show that there exists a continuous function f from X to $[0,1]$ such that $f \equiv 0$ in A and $f \equiv 1$ in B. This problem shows that there are many continuous functions in a metric space.

7. In class we showed that the set $P = \{f : \ f(x) > 0, \forall x \in [a,b]\}$ is an open set in $C[a,b]$. Show that it is no longer true if the norm is replaced by the L^1-norm. In other words, for each $f \in P$ and each $\varepsilon > 0$, there is some continuous g which is negative somewhere such that $\|g - f\|_1 < \varepsilon$.

8. Show that $[a,b]$ can be expressed as the intersection of countable open intervals. It shows in particular that countable intersection of open sets may not be open.

9. Optional. Show that every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals. Suggestion: Introduce an equivalence relation $x \sim y$ if x and y belongs to the same open interval in the open set and observe that there are at most countable many such intervals.

10. Let f be a function from (X,d) to (Y,ρ). Show that f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.